

SOME FAMILIES OF SUPER CONGRUENCES INVOLVING ALTERNATING MULTIPLE HARMONIC SUMS

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ABSTRACT. Let p be a prime. In this short note we study some families of super congruences involving the following alternating sums

$$\sum_{\substack{j_1+j_2+\cdots+j_n=2p^r \\ p \nmid j_1 j_2 \cdots j_n}} \frac{(-1)^{j_1+\cdots+j_n}}{j_1 \cdots j_n} \pmod{p^r},$$

which extend similar statements proved by Shen and Cai who treated the cases when $n = 4, 5$.

1. INTRODUCTION

Over the past quarter of a century, multiple zeta values (MZVs) and their various generalizations have been intensively studied by many mathematicians and physicists due to their important applications in quite a few different areas of mathematics and theoretical physics. Very recently, a finite version of MZVs has emerged which has been conjectured to be closely related to MZVs, see [14, Ch. 8]. These values are essentially the partial sums of the MZV series, commonly called the multiple harmonic sums, first truncated at different primes and then taken residues modulo the corresponding primes. Such congruences were first studied independently by the last author in [11, 12] and Hoffman in [3]. As an application, in [10] the last author proved, by using some special properties of the double harmonic sums, that for every odd prime p

$$\sum_{\substack{i+j+k=p \\ i,j,k>0}} \frac{1}{ijk} \equiv -2B_{p-3} \pmod{p}, \quad (1)$$

where B_k are Bernoulli numbers defined by the generating series

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Later, Ji gave an alternative simpler proof of (1) in [4] using some combinatorial techniques. Congruence (1) has since been generalized by either increasing the number of indices, changing the bound from p to multiples of p or a p -power, and/or considering the corresponding super congruences (see [1, 5, 8, 9, 13, 15]), or even allowing the alternating version of MHSs (see [6, 7]).

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Let \mathbb{N} and \mathbb{N}_0 be the set of positive integers and nonnegative integers, respectively. For any $n, d \in \mathbb{N}$ and $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$, we define the *multiple harmonic sums* (MHSs) by

$$\mathcal{H}_n(\mathbf{s}) := \sum_{n > k_1 > \dots > k_d > 0} \frac{1}{k_1^{s_1} \dots k_d^{s_d}}.$$

For example, $\mathcal{H}_{n+1}(1)$ is often called the n th harmonic number. We can also define the alternating version of these sums. For convenience, we denote by \bar{s} a *signed integer* for every $s \in \mathbb{N}$ such that $|\bar{s}| = s$ and $\text{sgn}(\bar{s}) = -1$. Let s_j be either a positive integer or a signed integer for all $j = 1, \dots, d$. Then the alternating MHS

$$\mathcal{H}_n(s_1, \dots, s_d) := \sum_{n > k_1 > \dots > k_d > 0} \frac{\text{sgn}(s_1)^{k_1} \dots \text{sgn}(s_d)^{k_d}}{k_1^{|s_1|} \dots k_d^{|s_d|}}.$$

For example, $\lim_{n \rightarrow \infty} \mathcal{H}_n(\bar{1})$ is just the well-known alternating harmonic series.

Our main results of this short note concern the following type of sums. Let \mathcal{P}_p be the set of positive integers not divisible by p . Define

$$\begin{aligned} Z_n(N, p) &:= \sum_{\substack{l_1 + l_2 + \dots + l_n = N \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \dots l_n}, \\ R_n^{(m)}(p^r) &:= \sum_{\substack{l_1 + l_2 + \dots + l_n = mp^r \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \dots l_n} = Z_n(mp^r, p) \quad \text{for } p \nmid m, \\ S_n^{(m)}(p^r) &:= \sum_{\substack{l_1 + l_2 + \dots + l_n = mp^r \\ p^r > l_1, \dots, l_n \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \dots l_n} \quad \text{for } p \nmid m. \end{aligned}$$

The primary goal of our study is to find nice and simple super congruences involving alternating sums defined as follows:

$$\sigma_n^{(b)}(N, p) := \sum_{\substack{l_1 + l_2 + \dots + l_n = N \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{(-1)^{l_1 + \dots + l_b}}{l_1 l_2 \dots l_n}.$$

We will reduce these congruences to those of $Z_n(N, p)$ whose special cases $R_n^{(m)}(p^r)$ are closely related to $S_n^{(m)}(p^r)$ by Proposition 2.3. These results are motivated by the recent work of Shen and Cai [7] who studied the above sums for $n = 3, 4$. In Theorem 3.4 we generalize this to arbitrary n by using $R_n^{(m)}(p^r)$ with $m = 1, 2$.

2. SOME USEFUL LEMMAS

We start with a formula expressing the sums $Z_n(N, p)$ in terms of a modified version of multiple harmonic sums.

Lemma 2.1. *Let $n, N \in \mathbb{N}$ and p be a prime. Then we have*

$$Z_n(N, p) = \frac{n!}{N} \sum_{\substack{1 \leq u_1 < \dots < u_{n-1} < N \\ u_1, u_2 - u_1, \dots, u_{n-1} - u_{n-2}, u_{n-1} \in \mathcal{P}_p}} \frac{1}{u_1 \dots u_{n-1}}. \quad (2)$$

Proof. First, noting that $l_1 + l_2 + \cdots + l_n = N$, we have

$$\begin{aligned} Z_n(N, p) &= \frac{1}{N} \sum_{\substack{l_1 + l_2 + \cdots + l_n = N \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{l_1 + l_2 + \cdots + l_n}{l_1 l_2 \cdots l_n} \\ &= \frac{n}{N} \sum_{N > l_1, \dots, l_{n-1} \in \mathcal{P}_p} \frac{1}{l_1 l_2 \cdots l_{n-1}}. \end{aligned}$$

Repeating this idea using the substitutions $u_j = l_1 + \cdots + l_j$ for $j = 1, \dots, n-1$, one readily proves equation (2). \square

Lemma 2.2. *Suppose $m, n, r \in \mathbb{N}$ and p is a prime with $p > n + 1$. Then we have*

$$S_n^{(m)}(p^{r+1}) \equiv (-1)^{m-1} \binom{n-2}{m-1} S_n^{(1)}(p^2) p^{r-1} \pmod{p^{r+1}}.$$

Proof. For all $n, a \in \mathbb{N}$, set

$$\gamma_n(a) := (-1)^{a+1} \frac{(a-1)!(n-1-a)!}{(n-1)!}.$$

By [5, Lemma 2.3], we have

$$\begin{aligned} S_n^{(m)}(p^{r+1}) &\equiv p \sum_{a=1}^{n-1} (-1)^{m-1} \binom{n-2}{m-1} \gamma_n(a) S_n^{(a)}(p^r) \pmod{p^{r+1}} \\ &\equiv (-1)^{m-1} \binom{n-2}{m-1} S_n^{(1)}(p^{r+1}) \pmod{p^{r+1}}. \end{aligned}$$

So the lemma follows from [5, (1.3)] which says

$$S_n^{(1)}(p^{r+1}) \equiv p S_n^{(1)}(p^r) \pmod{p^{r+1}}$$

for all $r \geq 2$. \square

Proposition 2.3. *Let $m, n, r \in \mathbb{N}$ with $r \geq 2$. Then we have*

$$\begin{aligned} R_n^{(m)}(p) &\equiv \sum_{a=1}^{n-1} \binom{m+n-a-1}{n-1} S_n^{(a)}(p) \pmod{p}, \\ R_n^{(m)}(p^r) &\equiv m \cdot S_n^{(1)}(p^2) p^{r-2} \pmod{p^r}. \end{aligned}$$

Proof. Let p be a prime number such that $p > n + 1$. For any n -tuples (l_1, \dots, l_n) of integers satisfying $l_1 + \cdots + l_n = mp^r$, $l_i \in \mathcal{P}_p$, $1 \leq i \leq n$, we rewrite them as

$$l_i = x_i p^r + y_i, \quad x_i \geq 0, \quad 1 \leq y_i < p^r, \quad y_i \in \mathcal{P}_p, \quad 1 \leq i \leq n.$$

Since

$$\left(\sum_{i=1}^n x_i \right) p^r + \sum_{i=1}^n y_i = mp^r,$$

there exists $1 \leq a < n$ such that

$$\begin{cases} x_1 + \cdots + x_n = m - a, \\ y_1 + \cdots + y_n = ap^r. \end{cases}$$

For $1 \leq a < n$, the equation $x_1 + \cdots + x_n = m - a$ has $\binom{m+n-a-1}{n-1}$ nonnegative integer solutions. Hence for all $r \geq 1$

$$\begin{aligned}
R_n^{(m)}(p^r) &= \sum_{\substack{l_1+\cdots+l_n=mp^r \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \cdots l_n} \\
&= \sum_{a=1}^{n-1} \sum_{x_1+\cdots+x_n=m-a} \sum_{\substack{y_1+\cdots+y_n=ap^r \\ y_i \in \mathcal{P}_p, y_i < p^r}} \frac{1}{(x_1 p^r + y_1) \cdots (x_n p^r + y_n)} \\
&\equiv \sum_{a=1}^{n-1} \binom{m+n-a-1}{n-1} S_n^{(a)}(p^r) \pmod{p^r} \\
&\equiv \sum_{a=1}^{n-1} \binom{m+n-a-1}{n-1} (-1)^{a-1} \binom{n-2}{a-1} S_n^{(1)}(p^2) p^{r-2} \pmod{p^r}.
\end{aligned}$$

by Lemma 2.2. Note that the penultimate step holds for $r = 1$ which implies the first equation in the proposition. However, the last step is valid only when $r \geq 2$. So the second congruence in the proposition follows immediately from

$$\sum_{a=1}^{n-1} (-1)^{a-1} \binom{n-2}{a-1} \binom{m+n-a-1}{n-1} = \sum_{a=1}^{n-1} \binom{a+1-n}{a-1} \binom{m+n-a-1}{m-a} = m$$

by the famous Chu–Vandermonde identity. \square

3. ALTERNATING SUMS

Recall that

$$\sigma_n^{(b)}(N, p) = \sum_{\substack{l_1+l_2+\cdots+l_n=N \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{(-1)^{l_1+\cdots+l_b}}{l_1 l_2 \cdots l_n}.$$

In this section, for each fixed $n \geq 4$, we will study some suitable linear combinations of $\sigma_n^{(b)}(N, p)$ for $b = 1, \dots, n-1$. To this end, for any $a \geq b \geq 0$, $d \geq 0$ and $\mathbf{s} = (s_1, \dots, s_d) \in \{1, \bar{1}\}^d$, we define

$$F_a^{(b)}(\mathbf{s}, N, p) := \sum_{\substack{N > i_1 > \cdots > i_d > l_1 + \cdots + l_a, \ l_1, \dots, l_a \in \mathcal{P}_p \\ i_1, i_1 - i_2, \dots, i_{d-1} - i_d, i_d - (l_1 + \cdots + l_a) \in \mathcal{P}_p}} \frac{\text{sgn}(s_1)^{i_1} \cdots \text{sgn}(s_d)^{i_d} (-1)^{l_1 + \cdots + l_b}}{i_1 \cdots i_d l_1 \cdots l_a}.$$

Then it is easy to see that if N is even then

$$N \sigma_n^{(b)}(N, p) = (n-b) F_{n-1}^{(b)}(\emptyset, N, p) + b F_{n-1}^{(n-b)}(\emptyset, N, p), \quad (3)$$

$$F_a^{(b)}(\mathbf{s}, N, p) = (a-b) F_{a-1}^{(b)}((\mathbf{s}, 1), N, p) + b F_{a-1}^{(a-b)}((\mathbf{s}, \bar{1}), N, p), \quad a \geq 1. \quad (4)$$

For $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$, put

$$X_m := (1_m), \quad Z_n := (\{\bar{1}\}, 1_{n-1}).$$

For $\mathbf{s} = (X_{w_1}, Z_{w_2}, \dots, Z_{w_l})$, we set $W_{\mathbf{s}} := (w_1, w_2, \dots, w_l)$ and $\text{len}(W_{\mathbf{s}}) := l$. For $\mathbf{s} = \emptyset$, define

$$A_{\emptyset} := 0, \quad B_{\emptyset} := 0, \quad W_{\emptyset} := (0), \quad P_{\emptyset} := b. \quad (5)$$

Otherwise, for $\mathbf{s} \neq \emptyset$, we define

$$A_{\mathbf{s}} := \sum_{2|i} w_i, \quad B_{\mathbf{s}} := \sum_{2|i} w_i, \quad P_{\mathbf{s}} := \begin{cases} a - b - A_{\mathbf{s}} & \text{if } 2 \nmid \text{len}(W_{\mathbf{s}}); \\ b - B_{\mathbf{s}} & \text{if } 2 \mid \text{len}(W_{\mathbf{s}}). \end{cases}$$

Finally, for all fixed $a \geq b \geq 0$ and $A, B \geq 0$, we put

$$C_{a,b}(A, B) = C(A, B) := \begin{cases} 1 & \text{if } A, B = 0; \\ (b)_B & \text{if } A = 0, B > 0; \\ (a - b)_A & \text{if } A > 0, B = 0; \\ (a - b)_A (b)_B & \text{if } A > 0, B > 0, \end{cases}$$

where $(x)_{\alpha} = x(x - 1) \cdots (x - \alpha + 1)$ is the Pochhammer symbol for the falling factorial.

Lemma 3.1. *Let $a, b \in \mathbb{N}_0$. Then for any fixed nonnegative integer $d \leq a$,*

$$F_a^{(b)}(\emptyset, N, p) = \sum_{\mathbf{s} \in \{1, \bar{1}\}^d} C_{a,b}(A_{\mathbf{s}}, B_{\mathbf{s}}) F_{a-A_{\mathbf{s}}-B_{\mathbf{s}}}^{(P_{\mathbf{s}})}(\mathbf{s}, N, p).$$

Proof. We will prove this by induction on d . If $d = 0$ then there is only one term in the sum corresponding to $\mathbf{s} = \emptyset$. Then the lemma holds by (5). Now let $d \geq 1$ and suppose the lemma is true when d is replaced by $d - 1$. Observe that any composition in $\{1, \bar{1}\}^d$ is produced by either $(\mathbf{s}, 1)$ or $(\mathbf{s}, \bar{1})$ for a unique $\mathbf{s} \in \{1, \bar{1}\}^{d-1}$. Further, it is easy to see that

$$\begin{aligned} (A_{(\mathbf{s}, 1)}, B_{(\mathbf{s}, 1)}) &= \begin{cases} (A_{\mathbf{s}} + 1, B_{\mathbf{s}}) & \text{if } 2 \nmid \text{len}(W_{\mathbf{s}}); \\ (A_{\mathbf{s}}, B_{\mathbf{s}} + 1) & \text{if } 2 \mid \text{len}(W_{\mathbf{s}}), \end{cases} \\ (A_{(\mathbf{s}, \bar{1})}, B_{(\mathbf{s}, \bar{1})}) &= \begin{cases} (A_{\mathbf{s}}, B_{\mathbf{s}} + 1) & \text{if } 2 \nmid \text{len}(W_{\mathbf{s}}); \\ (A_{\mathbf{s}} + 1, B_{\mathbf{s}}) & \text{if } 2 \mid \text{len}(W_{\mathbf{s}}). \end{cases} \end{aligned}$$

If $d < a$ and $2 \nmid \text{len}(W_{\mathbf{s}})$, then by (4)

$$\begin{aligned} & C(A_{\mathbf{s}}, B_{\mathbf{s}}) F_{a-A_{\mathbf{s}}-B_{\mathbf{s}}}^{(b-B_{\mathbf{s}})}(\mathbf{s}, N, p) \\ &= C(A_{\mathbf{s}}, B_{\mathbf{s}}) \left[(a - b - A_{\mathbf{s}}) F_{a-A_{\mathbf{s}}-B_{\mathbf{s}}-1}^{(b-B_{\mathbf{s}})}((\mathbf{s}, 1), N, p) + (b - B_{\mathbf{s}}) F_{a-A_{\mathbf{s}}-B_{\mathbf{s}}-1}^{(a-b-A_{\mathbf{s}})}((\mathbf{s}, \bar{1}), N, p) \right] \\ &= C(A_{\mathbf{s}} + 1, B_{\mathbf{s}}) F_{a-(A_{\mathbf{s}}+1)-B_{\mathbf{s}}}^{(P_{(\mathbf{s}, 1)})}((\mathbf{s}, 1), N, p) + C(A_{\mathbf{s}}, B_{\mathbf{s}} + 1) F_{a-A_{\mathbf{s}}-(B_{\mathbf{s}}+1)}^{(P_{(\mathbf{s}, \bar{1})})}((\mathbf{s}, \bar{1}), N, p) \\ &= C(A_{(\mathbf{s}, 1)}, B_{(\mathbf{s}, 1)}) F_{a-A_{(\mathbf{s}, 1)}-B_{(\mathbf{s}, 1)}}^{(P_{(\mathbf{s}, 1)})}((\mathbf{s}, 1), N, p) + C(A_{(\mathbf{s}, \bar{1})}, B_{(\mathbf{s}, \bar{1})}) F_{a-A_{(\mathbf{s}, \bar{1})}-B_{(\mathbf{s}, \bar{1})}}^{(P_{(\mathbf{s}, \bar{1})})}((\mathbf{s}, \bar{1}), N, p). \end{aligned}$$

If $d < a$ and $2 \mid \text{len}(W_{\mathbf{s}})$, then by (4) again

$$\begin{aligned} & C(A_{\mathbf{s}}, B_{\mathbf{s}}) F_{a-A_{\mathbf{s}}-B_{\mathbf{s}}}^{(a-b-A_{\mathbf{s}})}(\mathbf{s}, N, p) \\ &= C(A_{\mathbf{s}}, B_{\mathbf{s}}) \left[(b - B_{\mathbf{s}}) F_{a-A_{\mathbf{s}}-B_{\mathbf{s}}-1}^{(a-b-A_{\mathbf{s}})}((\mathbf{s}, 1), N, p) + (a - b - A_{\mathbf{s}}) F_{a-A_{\mathbf{s}}-B_{\mathbf{s}}-1}^{(b-B_{\mathbf{s}})}((\mathbf{s}, \bar{1}), N, p) \right] \\ &= C(A_{\mathbf{s}}, B_{\mathbf{s}} + 1) F_{a-A_{\mathbf{s}}-(B_{\mathbf{s}}+1)}^{(P_{(\mathbf{s}, 1)})}((\mathbf{s}, 1), N, p) + C(A_{\mathbf{s}} + 1, B_{\mathbf{s}}) F_{a-(A_{\mathbf{s}}+1)-B_{\mathbf{s}}}^{(P_{(\mathbf{s}, \bar{1})})}((\mathbf{s}, \bar{1}), N, p) \\ &= C(A_{(\mathbf{s}, 1)}, B_{(\mathbf{s}, 1)}) F_{a-A_{(\mathbf{s}, 1)}-B_{(\mathbf{s}, 1)}}^{(P_{(\mathbf{s}, 1)})}((\mathbf{s}, 1), N, p) + C(A_{(\mathbf{s}, \bar{1})}, B_{(\mathbf{s}, \bar{1})}) F_{a-A_{(\mathbf{s}, \bar{1})}-B_{(\mathbf{s}, \bar{1})}}^{(P_{(\mathbf{s}, \bar{1})})}((\mathbf{s}, \bar{1}), N, p). \end{aligned}$$

This finishes the induction proof of the lemma. \square

Corollary 3.2. *Let $a, b \in \mathbb{N}$ with $a \geq b$. For all $\mathbf{s} \in \{1, \bar{1}\}^a$, we have*

$$C_{a,b}(A_{\mathbf{s}}, B_{\mathbf{s}}) = \begin{cases} (a - b)! b! & \text{if } A_{\mathbf{s}} = a - b, B_{\mathbf{s}} = b; \\ 0 & \text{if } A_{\mathbf{s}} \neq a - b, B_{\mathbf{s}} \neq b. \end{cases}$$

Proof. It is easy to see that $A_{\mathbf{s}} + B_{\mathbf{s}} = |W_{\mathbf{s}}| = |\mathbf{s}| = a$. If $C_{a,b}(A_{\mathbf{s}}, B_{\mathbf{s}}) \neq 0$, then by its definition

$$a - b - A_{\mathbf{s}} + 1 > 0, b - B_{\mathbf{s}} + 1 = -(a - b - A_{\mathbf{s}}) + 1 > 0,$$

which imply that $A_{\mathbf{s}} = a - b, B_{\mathbf{s}} = b$ and $C_{a,b}(A_{\mathbf{s}}, B_{\mathbf{s}}) = (a - b)!b!$. \square

Corollary 3.3. *For all fixed $a \in \mathbb{N}$, we have*

$$\sum_{b=0}^a \binom{a}{b} F_a^{(b)}(\emptyset, 2N, p) = \frac{N}{a+1} Z_{a+1}(N, p).$$

Proof. By Corollary 3.2, $C(A_{\mathbf{s}}, B_{\mathbf{s}}) \neq 0$ for one and only one b for every $\mathbf{s} \in \{1, \bar{1}\}^a$. Thus,

$$\begin{aligned} \sum_{b=0}^a \binom{a}{b} F_a^{(b)}(\emptyset, 2N, p) &= \sum_{\mathbf{s} \in \{1, \bar{1}\}^a} a! F_0^{(0)}(\mathbf{s}, 2N, p) \\ &= a! \sum_{\substack{2N > i_1 > \dots > i_a > 0 \\ i_1, i_1 - i_2, \dots, i_{a-1} - i_a, i_a \in \mathcal{P}_p}} \frac{(1 + (-1)^{i_1}) \dots (1 + (-1)^{i_a})}{i_1 \dots i_a} \end{aligned}$$

As the term is nonzero only when all indices are even,

$$a! \sum_{\substack{2N > i_1 > \dots > i_a > 0 \\ i_1, i_1 - i_2, \dots, i_{a-1} - i_a, i_a \in \mathcal{P}_p}} \frac{(1 + (-1)^{i_1}) \dots (1 + (-1)^{i_a})}{i_1 \dots i_a} = a! \sum_{\substack{N > i_1 > \dots > i_a > 0 \\ i_1, i_1 - i_2, \dots, i_{a-1} - i_a, i_a \in \mathcal{P}_p}} \frac{1}{i_1 \dots i_a}.$$

We can now finish the proof of the corollary by using (2). \square

Theorem 3.4. *Let n, N be two positive integers. We have*

$$\sum_{b=1}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2N, p) = \frac{1}{2} Z_n(N, p) - Z_n(2N, p),$$

where $\alpha_{n,b} = 1$ except for $\alpha_{n,n/2} = 1/2$ when n is even. In particular, for all $r \in \mathbb{N}$ and primes p we have

$$\sum_{b=1}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2p^r, p) = \frac{1}{2} R_n^{(1)}(p^r) - R_n^{(2)}(p^r) \equiv -\frac{3}{2} S_n^{(1)}(p^2) p^{r-1} \pmod{p^r}.$$

Proof. For even N , we have $\sigma_n^{(b)}(N, p) = \sigma_n^{(n-b)}(N, p)$ and therefore we get

$$\begin{aligned} 2N \sum_{b=0}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2N, p) &= \frac{1}{2} \sum_{b=0}^n \binom{n}{b} 2N \sigma_n^{(b)}(2N, p) \\ &= \frac{1}{2} \sum_{b=0}^n \binom{n}{b} (n-b) F_{n-1}^{(b)}(\emptyset, 2N, p) + \frac{1}{2} \sum_{b=0}^n \binom{n}{b} b F_{n-1}^{(n-b)}(\emptyset, 2N, p) \end{aligned}$$

by (3). Using substitution $b \rightarrow n - b$ in the second sum, we get

$$2N \sum_{b=0}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2N, p) = \sum_{b=0}^n (n-b) \binom{n}{b} F_{n-1}^{(b)}(\emptyset, 2N, p)$$

$$= n \sum_{b=0}^{n-1} \binom{n-1}{b} F_{n-1}^{(b)}(\emptyset, 2N, p) = NZ_n(N, p),$$

by Corollary 3.3 with $a = n - 1$. Therefore,

$$\sum_{b=1}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2N, p) = \sum_{b=0}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2N, p) - \sigma_n^{(0)}(2N, p) = \frac{1}{2} Z_n(N, p) - Z_n(2N, p)$$

since $\sigma_n^{(0)}(2N, p) = Z_n(2N, p)$. The final congruence of the theorem follows easily from Proposition 2.3. This completes the proof of the theorem. \square

Corollary 3.5. *Let $n \in \mathbb{N}$ and p be a prime such that $p > n + 1$. Then we have*

$$\sum_{b=1}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2p, p) \equiv \begin{cases} \frac{n!}{2} B_{p-n} \pmod{p}, & \text{if } 2 \nmid n; \\ -\frac{n!}{2} \sum_{a+b=n, a, b \geq 3} \frac{B_{p-a} B_{p-b}}{ab} \pmod{p}, & \text{if } 2 \mid n. \end{cases}$$

Proof. This follows easily from Theorem 3.4, [15, Main Theorem], [5, Lemma 3.5 and Corollary 3.6] (for n odd) and [9, Theorem 1 and Corollary 1] (for n even). \square

Corollary 3.6. *Let $r \in \mathbb{N}$ and $p > 4$ be a prime. We have*

$$\sigma_4^{(1)}(2p^r, p) + 3\sigma_4^{(2)}(2p^r, p) \equiv 0 \pmod{p^r}, \quad (6)$$

$$\sigma_5^{(1)}(2p^r, p) + 2\sigma_5^{(2)}(2p^r, p) \equiv 6B_{p-5}p^{r-1} \pmod{p^r}. \quad (7)$$

Proof. It follows from [9, Theorem 1], [13, Theorem 1.1], [15, Main Theorem], and [8, Theorem 2] that

$$S_4^{(1)}(p^2) \equiv 0, \quad S_5^{(1)}(p^2) \equiv -20B_{p-5}p \pmod{p^2}.$$

So Theorem 3.4 yields the corollary immediately. \square

In fact, this note was motivated by Shen and Cai's proof of (6) and a finer version of (7) in [6]. Now it follows from [9, Theorem 4] and [5, Theorem 1.1] that

$$S_6^{(1)}(p^2) \equiv -\frac{20}{3}B_{p-3}^2p, \quad S_7^{(1)}(p^2) \equiv -504B_{p-7}p \pmod{p^2},$$

and, by similar computation (see [2] for details)

$$S_8^{(1)}(p^2) \equiv -\frac{1792}{5}B_{p-3}B_{p-5}p, \quad S_9^{(1)}(p^2) \equiv -\frac{32}{3}(2283B_{p-9} + 7B_{p-3}^3)p \pmod{p^2}.$$

Therefore, by Theorem 3.4, modulo p^r ($r \geq 2$), we have

$$6\sigma_6^{(1)}(2p^r, p) + 15\sigma_6^{(2)}(2p^r, p) + 10\sigma_6^{(3)}(2p^r, p) \equiv 10B_{p-3}^2p^{r-1},$$

$$7\sigma_7^{(1)}(2p^r, p) + 21\sigma_7^{(2)}(2p^r, p) + 35\sigma_7^{(3)}(2p^r, p) \equiv 756B_{p-7}p^{r-1},$$

$$8\sigma_8^{(1)}(2p^r, p) + 28\sigma_8^{(2)}(2p^r, p) + 56\sigma_8^{(3)}(2p^r, p) + 35\sigma_8^{(4)}(2p^r, p) \equiv \frac{2688}{5}B_{p-3}B_{p-5}p^{r-1},$$

$$9\sigma_9^{(1)}(2p^r, p) + 36\sigma_9^{(2)}(2p^r, p) + 84\sigma_9^{(3)}(2p^r, p) + 126\sigma_9^{(4)}(2p^r, p) \equiv 16(2283B_{p-9} + 7B_{p-3}^3)p^{r-1}.$$

By combining Theorem 3.4 and the numerical results of $S_n^{(1)}(p^2)$ obtained in [2], one can derive similar explicit formulas for all $n \geq 10$ easily.

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